Interest Rate Theory in the Presence of Multiple Yield Curves – An FX-like Approach

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3rd Imperial - ETH Workshop on Mathematical Finance

March 5, 2015

Outline

FX-analogy by Jarrow & Turnbull

We consider the following zero-coupon bonds with maturity T:

Type	non-defaultable	defaultable
t-Price	$P(t,T)$	$\widetilde{P}(t,T)$
Payoff	$P(T,T)=1$	$0<\widetilde{P}(T,T)\leq 1$ (random)

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Paradigm (Jarrow & Turnbull 1991)

$\widetilde{P}(t,T) = S_t Q(t,T)$ may be interpreted as conversion of foreign default-free counterparts. $S_t := \widetilde{P}(t,t)$ is referred to as **recovery rate** or **spot FX rate** at time t.

FX-analogy by Jarrow & Turnbull

We consider the following zero-coupon bonds with maturity T:

Type	non-defaultable	defaultable	non-defaultable
Currency	domestic	domestic	foreign
t-Price	$P(t,T)$	$\widetilde{P}(t,T)$	$Q(t,T) := \frac{\widetilde{P}(t,T)}{\widetilde{P}(t,t)}$
Payoff	$P(T,T) = 1$	$0 < \widetilde{P}(T,T) \leq 1$ (random)	$Q(T,T) = 1$

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$r = (r_t)_{t \geq 0}$	short rate process w.r.t. EMM $\mathbb{Q}$,
$N = (N_t)_{t \geq 0}$	Cox-process with intensity $\lambda = (\lambda_t)_{t \geq 0}$ and jumps at the random times $\{\tau_i\}_{i \in \mathbb{N}}$,
$s = (s_t)_{t \geq 0}$	$(0,1)$-valued loss quota process with first moments $\bar{s}_t := E_{\mathbb{Q}}\big[s_t\big]$,
$dS_t = -S_{t-} s_t \, dN_t, \;\; S_0 = 1$	recovery rate process.

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Theorem (Duffie-Singleton, Schönbucher)

Under suitable technical assumptions, we have for all $0 \leq t \leq T < \infty$

$$\widetilde{P}(t,T) = \underbrace{\left(\prod_{\tau_i \leq t}\left(1 - s_{\tau_i}\right)\right)}_{=S_t}\underbrace{E_{\mathbb{Q}}\left[e^{-\int_t^T r_u + \bar{s}_u\lambda_u\,du}\middle|\mathcal{F}_t\right]}_{=Q(t,T)}.$$

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- The FX-like approach allows for interpretations that comply with the common economic intuition, e.g., the differentiation between liquidity squeezes and true default events.
- The recovery rate admits a natural economic interpretation by characterising to what extent the related party is able to meet its imminent financial obligations. However, what is a meaningful recovery rate in time instances in which no payments are due?

Outline

The General FX-like Setting

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Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be a filtered probability space satisfying the usual conditions. We consider three $\mathbb{F}$-adapted series of zero-coupon bond prices, where the properties on the right-hand side shall hold a.s. for all maturities $T \geq 0$.

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$\{P(t,T)\}_{0 \leq t \leq T < \infty}$	Domestic non-defaultable zero-coupon bonds with payoff $P(T,T) = 1$.
$\{\widetilde{P}(t,T)\}_{0 \leq t \leq T < \infty}$	Domestic defaultable zero-coupon bonds with a random payoff $0 < \widetilde{P}(T,T) \leq 1$.
$\{Q(t,T)\}_{0 \leq t \leq T < \infty}$	Synthetic foreign non-defaultable zero-coupon bonds satisfying the relation $$Q(t,T) = \frac{\widetilde{P}(t,T)}{\widetilde{P}(t,t)}.$$

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Having the *Fundamental Theorem of Asset Pricing* for frictionless markets in mind, we assume that there exists an equivalent local martingale measure (ELMM) $\mathbb{Q} \approx \mathbb{P}$ such that the discounted processes

$$\left(\frac{P(t,T)}{B_t}\right)_{0 \leq t \leq T}, \qquad \left(\frac{S_t Q(t,T)}{B_t}\right)_{0 \leq t \leq T} = \left(\frac{\widetilde{P}(t,T)}{B_t}\right)_{0 \leq t \leq T}$$

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Corresponding HJM-framework: Amin and Jarrow Economy, [AJ1991]

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In multi-currency settings, the ratio

$$F(t,T) := \frac{\widetilde{P}(t,T)}{P(t,T)} = S_t \frac{Q(t,T)}{P(t,T)}$$

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If $\mathbb{Q}$ is an EMM and $\mathbb{Q}^T$ denotes the induced T-forward measure associated with the numéraire $\big(P(t, T)\big)_{0 \le t \le T}$, then $\big(F(t, T)\big)_{0 \le t \le T}$ defines a $\mathbb{Q}^T$-martingale.

Arbitrage-Free Interpolation

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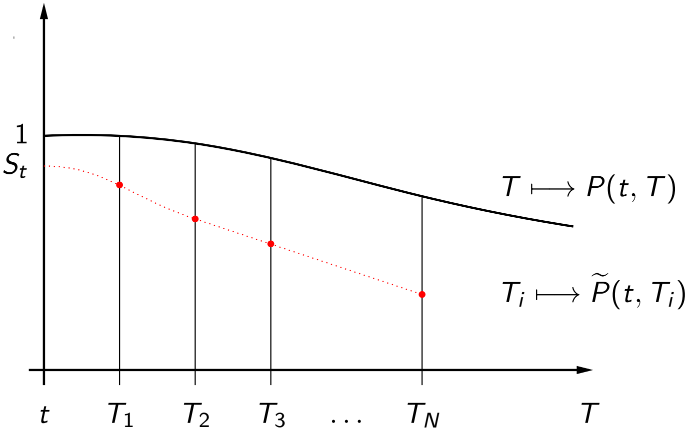

Arbitrage-Free Interpolation

We assume that a intermittent but arbitrage-free interest rate framework is given w.r.t. EMM $\mathbb{Q}$:

$B = (B_t)_{t \geq 0}$	bank account numéraire,
$[t, \infty) \longrightarrow \mathbb{R}, T \longmapsto P(t, T)$	comprehensive term structure for non-defaultable bonds for any $t \geq 0$,
$0 = T_0 < T_1 < \ldots < T_N = T^*$	discrete tenor structure,
$\left(\widetilde{P}(t, T_i)\right)_{0 \leq t \leq T_i}$	inferrable defaultable bond prices for $i = 1, 2, \ldots, N$.

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Objective: Complementing this setting to an enhanced credit risk framework by interpolating the discrete defaultable term structure in the maturity dimension.

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Let

$$k(T) := \max\left\{ i = 1, 2, \ldots, N \,\middle|\, T_{i-1} < T \right\}$$

be the index of the next upcoming gridpoint and $\vartheta : \mathbb{T} \longrightarrow [0, 1]$ be any (deterministic) RCLL function with

$$\lim_{\delta \to 0+} \vartheta(T_i + \delta) = 1, \qquad \lim_{\delta \to 0+} \vartheta(T_{i+1} - \delta) = 0$$

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We make for all $T \in [0, T^*]$ the ansatz

$$S_T := \vartheta(T) \frac{1}{P\big(T_{k(T)-1}, T\big)} S_{T_{k(T)-1}} + \big(1 - \vartheta(T)\big) P\big(T, T_{k(T)}\big) F\big(T, T_{k(T)}\big).$$

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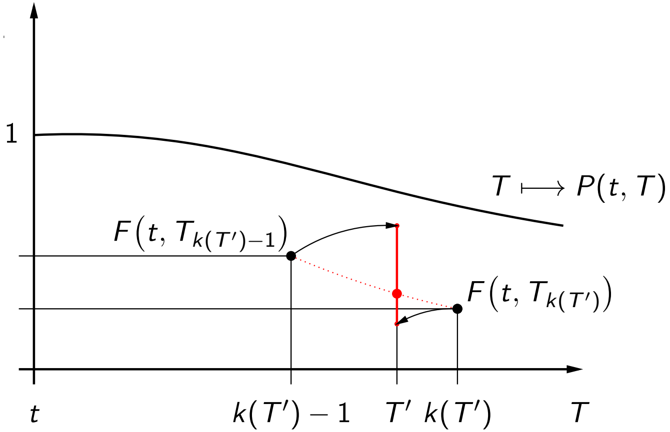

$$F(t, T') := \vartheta(T')\frac{P(t, T_{k(T')-1})}{P(t, T')}F\big(t, T_{k(T')-1}\big) + \big(1 - \vartheta(T')\big)\frac{P(t, T_{k(T')})}{P(t, T')}F\big(t, T_{k(T')}\big).$$

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More precisely,

$$F(t,T) := \begin{cases} \vartheta(T)\frac{P(t,T_{k(T)-1})}{P(t,T)}F\big(t,T_{k(T)-1}\big) + \big(1-\vartheta(T)\big)\frac{P(t,T_{k(T)})}{P(t,T)}F\big(t,T_{k(T)}\big) & \text{, if } t \leq T_{k(T)-1}, \\ \vartheta(T)\frac{1}{P(T_{k(T)-1},T)}S_{T_{k(T)-1}} + \big(1-\vartheta(T)\big)\frac{P(t,T_{k(T)})}{P(t,T)}F\big(t,T_{k(T)}\big) & \text{, if } t > T_{k(T)-1}. \end{cases}$$

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Proposition

Let the intermittent interest rate framework be given. If one follows the proposed interpolation scheme, then $\big(F(t,T)\big)_{0\leq t\leq T}$ forms a $\mathbb{Q}^T$-martingale for each $T \in [0, T^*]$.

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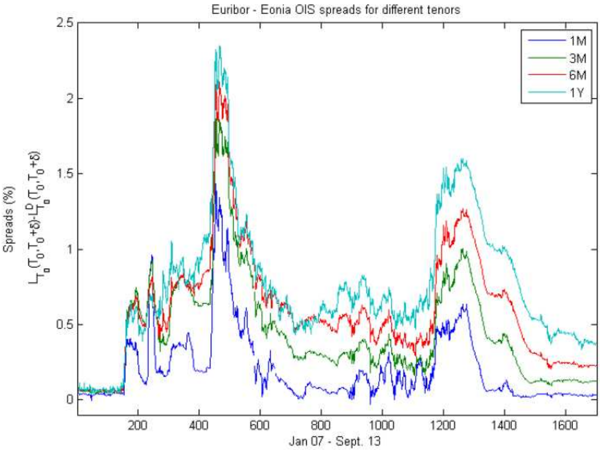
Proposition

Let the intermittent interest rate framework be given. If one follows the proposed interpolation scheme, then $\big(F(t,T)\big)_{0\leq t\leq T}$ forms a $\mathbb{Q}^T$-martingale for each $T \in [0, T^*]$.

- Remarkably, the scheme implies arbitrage-free dynamics for the forward recovery rate and works irrespective of the underlying distributions.
- It provides a very nice option of what a meaningful (forward) recovery rate may be in time instances in which no payments are due.

Outline

Modelling of the Interbank Market



[CFG2014] provides a general HJM-framework for multiple yield curve modelling. Each Libor rate to a certain tenor has its own foreign market.

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- Consistent recalibration (CRC) models, [HSTW2015]

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